

The curvature perturbation in a box

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The stochastic properties of cosmological perturbations are best defined through the Fourier expansion in a finite box. I discuss the reasons for that with reference to the curvature perturbation, and explore some issues arising from it.

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I. INTRODUCTION

Strong observational constraints on the primordial curvature perturbation ζ make it an important discriminator between models of the very early Universe. After smoothing relevant quantities on the shortest scale of interest, ζ may be defined as the fractional perturbation $\delta a/a$ of the locally-defined scale factor $a(\mathbf{x}, t)$. Equivalently, it is the perturbation in the number $N(\mathbf{x}, t)$ of e -folds of expansion, starting on an initial ‘flat’ slice of spacetime (where $\delta a = 0$) and ending on a slice of uniform energy density at time t . The spacetime threads of constant \mathbf{x} are taken to be comoving.

Any choice of the initial ‘flat’ slice will do, as long as the smoothing scale is outside the horizon at that stage, because the expansion going from one ‘flat’ slice to another is uniform. The final slice is to be located before the smoothing scale re-enters the horizon, and on cosmological scales it should be late enough that ζ has settled down to the final time-independent value that is constrained by observation. With the smoothing scale outside the horizon, the evolution at each point is expected to be that of some unperturbed universe (the separate universe assumption [1, 2, 3, 4, 5]).

During inflation, one or more of the scalar field perturbations is supposed to be practically massless during inflation ($m^2 \ll H^2$). For these ‘light’ fields, the vacuum fluctuation is promoted to a classical perturbation as each scale leaves the horizon. To calculate δN one smooths the light field perturbations on a comoving scale shorter than any of interest. A few Hubble times after the scale leaves the horizon the light fields have classical perturbations. Their values at this epoch are supposed to determine $N(\mathbf{x}, t)$ and hence ζ . Taking for simplicity just one field σ we have

$$\zeta(\mathbf{x}, t) = \delta N(\sigma(\mathbf{x})) \quad (1)$$

$$\equiv N(\sigma(\mathbf{x})) - N(\bar{\sigma}) \quad (2)$$

$$= N' \delta \sigma(\mathbf{x}) + \frac{1}{2} N'' \delta \sigma^2(\mathbf{x}) + \dots, \quad (3)$$

where

$$\delta \sigma(\mathbf{x}) \equiv \sigma(\mathbf{x}) - \bar{\sigma}, \quad (4)$$

and

$$N' \equiv \left. \frac{dN}{d\sigma} \right|_{\bar{\sigma}}, \quad N'' \equiv \left. \frac{d^2 N}{d\sigma^2} \right|_{\bar{\sigma}} \quad (5)$$

and so on.

The final expression for ζ is a power series in the field perturbation defined on the initial slice. The unperturbed value of σ is defined as its spatial average of each field, which simplifies the analysis.

We are interested in the Fourier components of ζ :

$$\zeta_{\mathbf{k}} = \int e^{-i\mathbf{k} \cdot \mathbf{x}} \zeta(\mathbf{x}) d^3 x, \quad (6)$$

and similarly for $\delta \sigma$. The perturbation $\delta \sigma_{\mathbf{k}}$ is supposed to originate from the vacuum fluctuation, and can be considered [6] as classical starting a few Hubble times after the epoch of horizon exit $k = aH \equiv \dot{a}$, where a is the scale factor. At this stage the correlators of $\delta \sigma_{\mathbf{k}}$ on scales not too far outside the horizon can easily be calculated using perturbative quantum field theory. Assuming Lorentz invariance and a quadratic kinetic term, $\delta \sigma$ is almost gaussian. The subsequent separate-universe evolution of $\delta \sigma(\mathbf{x})$ then gives the correlators of $\delta \sigma$ at the initial epoch. Finally, Eq. (3) gives the correlators of ζ which can be compared with observation.

To avoid assumptions about the unknowable Universe very far beyond the present horizon H_0^{-1} , this calculation should be done within a comoving box, whose present size L is not too much bigger than H_0^{-1} . This situation has not been discussed much in the literature for two reasons. First, with the box size not too much bigger than H_0^{-1} (minimal box) the linear term of Eq. (3) dominates. Then correlators are practically independent of the box size, so that only $\bar{\sigma}$ need be specified. Second, if σ is the inflaton in single-field slow-roll inflation and we use a minimal box, then $\bar{\sigma}$ can be calculated from the inflation model.

On the other hand, σ may have nothing to do with inflation, as in the curvaton model [7]. Also, we should know how to handle the dependence on box size as a matter of principle. In this note I look at some of the issues raised when one takes seriously the box size, developing some earlier work [5, 8].

II. CORRELATORS OF ζ

A model of the early Universe will predict, not $\zeta(\mathbf{x})$ itself but correlators, $\langle \zeta(\mathbf{x}) \zeta(\mathbf{y}) \rangle$ and so on. The $\langle \rangle$ is an ensemble average, which in the inflationary cosmology becomes a Heisenberg-picture vacuum expectation value.

As the vacuum is invariant under translations, so are the correlators.

The vacuum is also invariant under rotations, and so are the correlators.¹ Invariance under translations and rotations constrains the form of the correlators. One usually considers only the two, three and four-point correlators.

The two-point correlator is

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_{\zeta}(k), \quad (7)$$

where P_{ζ} is the spectrum. It is useful to define $\mathcal{P}_{\zeta} \equiv (k^3/2\pi^2)P_{\zeta}$, also called the spectrum. The mean-square of ζ is

$$\langle \zeta^2(\mathbf{x}) \rangle = \int_{L^{-1}}^{k_{\max}} \mathcal{P}_{\zeta}(k) \frac{dk}{k}, \quad (8)$$

where k_{\max} is the scale leaving the horizon at the initial epoch, and the infrared cutoff is provided by the box size. On cosmological scales observation gives $\mathcal{P}_{\zeta} = (5 \times 10^{-5})^2$ with little scale dependence.

The three-point correlator is

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_{\zeta}(k_1, k_2, k_3), \quad (9)$$

where B_{ζ} is the bispectrum. The connected contribution to the four-point correlator is

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_c = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) T_{\zeta}. \quad (10)$$

The trispectrum T_{ζ} is a function of six scalars, defining the quadrilateral formed by $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4\}$. There is also a disconnected contribution:

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_d = \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle \langle \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle + \text{perms.} \quad (11)$$

For any correlator, the connected correlator is the one that comes with an overall delta function. If the two-point correlator is the only connected one, ζ is said to be Gaussian. Data are at present consistent with the hypothesis that ζ is perfectly gaussian, but they might not be in the future.

As a result of translation invariance, the ensemble averages $\langle \zeta(\mathbf{x}) \zeta(\mathbf{y}) \rangle$ etc. can be regarded as a spatial average with fixed $\mathbf{x} - \mathbf{y}$. This is the ergodic theorem, whose proof for correlators is very simple. Considering the two-point correlator the definition of P_{ζ} gives

$$\langle \zeta(\mathbf{y}) \zeta(\mathbf{x} + \mathbf{y}) \rangle = \int P(k) e^{i\mathbf{k} \cdot \mathbf{x}} d^3k. \quad (12)$$

On the other hand, the spatial average is

$$\begin{aligned} & L^{-3} \int \zeta(\mathbf{y}) \zeta(\mathbf{x} + \mathbf{y}) d^3y \\ &= L^{-3} (2\pi)^{-6} \int \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} e^{i[\mathbf{k} \cdot \mathbf{y} + \mathbf{k}' \cdot (\mathbf{x} + \mathbf{y})]} d^3y d^3k d^3k' \\ &= L^{-3} (2\pi)^{-3} \int \delta^3(\mathbf{k} + \mathbf{k}') \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} e^{i\mathbf{k} \cdot \mathbf{x}} d^3k d^3k'. \end{aligned} \quad (13)$$

In the final expression, $\zeta_{\mathbf{k}} \zeta_{\mathbf{k}'}$ can be replaced by its ensemble average $\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle$, because each volume element $d^3k d^3k'$ can contain an arbitrary large number of points. (Remember that we are working in a finite box so that the possible momenta form a cubic lattice in \mathbf{k} -space. Within a cell, the Fourier coefficients are uncorrelated because of the delta functions in Eqs. (7), (9), and (10) and so on.) Writing $\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle$ in terms of $P(k)$ and using the rule $[\delta^3(\mathbf{k} - \mathbf{k}')]^2 = L^3 \delta^3(\mathbf{k} - \mathbf{k}')$, we see that the spatial average (13) is indeed equal to the ensemble average (12). reproduces the ensemble average, and the same argument works for higher correlators too.

From this proof it is clear that the ergodic theorem works in a finite box, with the usual proviso that the box size is bigger than scales of interest. Its proof relies just on translation invariance, which makes mathematical sense in the finite box because of the periodic boundary condition.

III. THE CORRELATORS OF $\delta\sigma$

The correlators of $\delta\sigma$ have the same form as those of ζ . In particular

$$\langle \sigma_{\mathbf{k}} \sigma_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_{\sigma}(k) \quad (14)$$

$$\langle \sigma_{\mathbf{k}_1} \sigma_{\mathbf{k}_2} \sigma_{\mathbf{k}_3} \sigma_{\mathbf{k}_4} \rangle_d = \langle \sigma_{\mathbf{k}_1} \sigma_{\mathbf{k}_2} \rangle \langle \sigma_{\mathbf{k}_3} \sigma_{\mathbf{k}_4} \rangle + \text{perms.} \quad (15)$$

The chosen box should be well inside the horizon at the beginning of inflation. After a few Hubble times, the universe inside the box is then expected [9] to become practically homogeneous and isotropic at the classical level. The conversion of the vacuum fluctuation into a classical perturbation $\delta\sigma$ begins when the box leaves the horizon at the epoch $k = aH$.

We observe scales $k \gtrsim H_0$ where H_0 is the present value of the Hubble parameter. Taking inflation to be almost exponential there are at most 60 or so Hubble times between horizon exit for the scale $k = H_0$ and the end of inflation. Of most interest is the perturbation on cosmological scales, which leave the horizon during the first 10 or so Hubble times. Smaller scale perturbations could also be of interest, for example to form black holes at the end of inflation.

We can distinguish between two kinds of box. A *minimal box*, for which $\ln(LH_0)$ is not too big, will leave the horizon just a few e -folds before the observable Universe leaves the horizon, and hence not too many Hubble times before all observable scales leave the horizon. A *super-large box* on the other hand, with very large $\ln(LH_0)$,

¹ In considering translations and rotations one takes \mathbf{x} to denote Cartesian coordinates defined in the unperturbed universe. In the perturbed universe these same coordinates cover the curved spacetime slice of fixed t , in a way that is defined by the threading which is here taken to be comoving.

would leave the horizon very many e -folds before the observable universe. We will see how to calculate things in a minimal box, and point to the difficulties that are encountered if one considers instead a super-large box.

With the minimal box, $\delta\sigma$ remains small at least while cosmological scales leave the horizon. On the usual assumption that σ is canonically normalized, the perturbation $\delta\sigma$ generated from the vacuum is then almost gaussian on scales not too far outside the horizon. Its spectrum is [10]

$$\mathcal{P}_\sigma(k) \simeq (H_k/2\pi)^2, \quad (16)$$

where H_k is the Hubble parameter at horizon exit. The bispectrum [11] and trispectrum [12] of $\delta\sigma$ have been calculated, and are suppressed by slow-roll factors. (These are $|\dot{H}/H^2| \ll 1$, needed for inflation, and parameters involving derivatives of the potential $V(\sigma)$ which have to be small to justify the perturbative quantum field theory calculation.)

As we saw earlier, the correlation functions defined by the spectrum, bispectrum and so on may be defined as spatial averages within the box. With a minimal box it is reasonable to assume that these are the same as the spatial averages within the region of size H_0^{-1} around us, that can actually be observed.

If we consider instead a super-large box several difficulties arise. We have to assume that enough inflation took place for the box to exist, and we need to understand the field theory starting from the era when the box leaves the horizon. When observable scales leave the horizon, the perturbation that has already been generated on larger scales may be large, which would complicate the calculation of $\delta\sigma$. And even when the calculation has been performed, the spatial averages represented by the correlators may have nothing to do with the spatial averages that are actually observed.

IV. THE MEAN VALUE $\bar{\sigma}$

A. With σ the inflaton

Now we come to the mean value $\bar{\sigma}$, of the field within the box. If σ is the inflaton in a single-field slow-roll inflation model, and we use a minimal box, then $\bar{\sigma}$ when N e -folds of inflation remain is given by

$$N(\sigma) = M_{\text{P}}^2 \int_{\sigma_{\text{end}}}^{\bar{\sigma}} \frac{V}{V'} d\sigma. \quad (17)$$

This follows from $dN = H dt$ and the slow-roll approximations

$$3H\dot{\sigma} = -V' \quad (18)$$

$$3H^2 M_{\text{P}}^2 = V \quad (19)$$

$$\dot{H}/H^2 = -\dot{\sigma}^2. \quad (20)$$

The inflation model will give the field σ_{end} at the end of inflation, and the post-inflationary cosmology determines N when a given scale leaves the horizon, with $N \sim 50$ or so for cosmological scales in the usual cosmology.

In this very special case, ζ becomes time-independent soon after horizon exit. As a result the small perturbation $\zeta = \delta N \sim 10^{-5}$ is then given in terms of the potential and its derivatives [11, 13]:

$$\begin{aligned} \zeta = & \frac{1}{M_{\text{P}}^2} \frac{V}{V'} \delta\sigma + \frac{1}{2} (2\eta - \epsilon) \left(\frac{1}{M_{\text{P}}^2} \frac{V}{V'} \delta\sigma \right)^2 \\ & + \frac{1}{6} (2\epsilon\eta - 2\eta^2 + \xi^2) \left(\frac{1}{M_{\text{P}}^2} \frac{V}{V'} \delta\sigma \right)^3 + \dots, \end{aligned} \quad (21)$$

where $2\epsilon = M_{\text{P}}^2 (V'/V)^2$, $\eta = M_{\text{P}}^2 V''/V$ and $\xi^2 = M_{\text{P}}^4 V'''V'/V^2$. With a minimal box the linear term dominates, and taking the initial slice to be soon after horizon exit on a given scale we then arrive at the famous result

$$\mathcal{P}_\zeta \simeq \left(\frac{1}{M_{\text{P}}^2} \frac{V}{V'} \frac{H}{2\pi} \right)^2, \quad (22)$$

where the right hand side is evaluated at horizon exit. The bispectrum and trispectrum can also be calculated, as described after Eq. (44).

Going to a super-large box, none of this may work. As we noted earlier, $\delta\sigma$ might be big which would invalidate the calculation of its stochastic properties. Also, δN might be big and then Eq. (17) would apply only to the average of N within the box (σ_{end} then being a spatial average), which might have little to do with the situation in the observable Universe.

B. With σ a curvaton-type field

Now suppose instead that σ has nothing to do with the inflation dynamics, as is typically (though not inevitably [14]) the case in the curvaton model.² Then at last we encounter a case where it may be useful to consider a super-large box, using what is often called the stochastic approach to the evolution of perturbations [15].

In the stochastic approach one takes spacetime to be unperturbed with constant H (de Sitter spacetime). One smooths σ on a practically-fixed scale $(1+b)H$ with $0 < b \ll 1$ a constant, and considers the probability $F(t, \sigma) d\sigma$ that $\sigma(\mathbf{x}, t)$ lies within a given interval. It satisfies the Fokker-Planck equation

$$\frac{\partial F}{\partial t} = \frac{V'}{3H} \frac{\partial F}{\partial \sigma} + \frac{H^3}{8\pi^2} \frac{\partial^2 F}{\partial \sigma^2}, \quad (23)$$

² For slow-roll inflation this corresponds to V' being much less than the corresponding quantity for the inflaton but we are not making any assumption about the inflation model.

This equation, applying to any slow-rolling field, corresponds to the Langevin equation describing the classical evolution plus the random walk $\pm H/2\pi$ per Hubble time coming from the creation of the classical perturbation from the vacuum fluctuation.

The point now is that the probability distribution may lose memory of the initial condition. In particular, if \dot{H}/H^2 is sufficiently small, F will settle down to [16]

$$F = \text{const} \exp\left(-\frac{8\pi^2}{3H^4}V(\sigma)\right), \quad (24)$$

More generally, one can handle a significant variation of H within a given inflation model (see for instance [17, 18]).

Reverting now to a minimal box, the probability distribution F should apply to $\bar{\sigma}$ if the initial epoch is taken to be soon after the shortest cosmological scale leaves the horizon, since the box size is then not too many e -folds bigger than the Hubble scale at the initial epoch. In any case, one could calculate the probability distribution of $\bar{\sigma}$ by going back to the Langevin equation. The very simplest case arises if σ is a pseudo Nambu-Goldstone boson (PNBG) with V' negligible. Then it is defined only in some interval $0 < \sigma < f$, and the noise term gives σ an equal probability of being anywhere in the interval.

Finally, given a probability distribution for $\bar{\sigma}$ one may suppose that the actual value for a minimal box around the observable Universe is not too far from the most probable value (or from say the mean-square if we are not dealing with a PNGB). Of course this final step is speculative and may be modified by environmental considerations. Still, it seems that in this case the use of a super-large box *purely to get a handle on the likely value of $\bar{\sigma}$ within a minimal box* may be helpful.

V. CALCULATING THE CORRELATORS OF ζ

Using the convolution theorem,

$$(\delta\sigma^2)_{\mathbf{k}} = \int \delta\sigma_{\mathbf{q}} \delta\sigma_{\mathbf{k}-\mathbf{q}} d^3q, \quad (25)$$

Eq. (3) determines the correlators of ζ in terms of those of $\delta\sigma$. There is a sum of terms and the calculation is best done using Feynman-like graphs [8, 19, 20]. A complete set of rules for constructing the graphs is given in [20]. A graph with n loops involves an integration of n momenta, while a tree-level involves no integration.

Here I recall the estimates of the correlators made in [5, 8] for the quadratic truncation of Eq. (3). To get an idea of what happens with higher terms included, I then consider the cubic truncation. In both cases I take $\delta\sigma$ to be gaussian. (See [19] for a loop contribution involving the bispectrum of $\delta\sigma$.)

A. Quadratic truncation

Truncating the field expansion after the quadratic term we have

$$\zeta(\mathbf{x}) = N'\delta\sigma(\mathbf{x}) + \frac{1}{2}N''\delta\sigma^2(\mathbf{x}). \quad (26)$$

There are tree-level and one-loop contributions to the correlators of ζ :

$$P_{\zeta}^{\text{tree}} = N'^2 P_{\sigma}(k) \quad (27)$$

$$P_{\zeta}^{\text{loop}} = \frac{N''^2}{(2\pi)^3} \int_{L^{-1}} d^3p P_{\sigma}(p) P_{\sigma}(|\mathbf{p} - \mathbf{k}|) \quad (28)$$

$$B_{\zeta}^{\text{tree}} = 2N'^2 N'' P_{\sigma}(k_1) P_{\sigma}(k_2) + \text{cyclic} \quad (29)$$

$$B_{\zeta}^{\text{loop}} = \frac{N''^3}{(2\pi)^3} \int_{L^{-1}} d^3p P_{\sigma}(p) P_{\sigma}(p_1) P_{\sigma}(p_2) \quad (30)$$

$$T_{\zeta}^{\text{tree}} = N'^2 N''^2 P_{\sigma}(k_1) P_{\sigma}(k_2) P_{\sigma}(k_{14}) + 23\text{perms.} \quad (31)$$

$$T_{\zeta}^{\text{loop}} = \frac{1}{8} \frac{N''^4}{(2\pi)^3} \int_{L^{-1}} d^3p P_{\sigma}(p) P_{\sigma}(p_1) P_{\sigma}(p_2) P_{\sigma}(p_{24}) + 23\text{perms.} \quad (32)$$

We have defined $p_1 \equiv |\mathbf{p} - \mathbf{k}_1|$, $p_2 \equiv |\mathbf{p} + \mathbf{k}_2|$, $p_{24} \equiv |\mathbf{p} + \mathbf{k}_{24}|$ and $k_{14} = |\mathbf{k}_1 + \mathbf{k}_4|$.

The 24 terms in Eq. (31) are actually 12 pairs of identical terms, and the 24 terms in Eq. (32) are actually 3 octuplets of identical terms. The tree-level contribution to the bispectrum was given in [22] and the tree-level contribution to the trispectrum was given in [8] (see also [21]). The loop contributions to the spectrum, bispectrum and trispectrum were given in [8], using \mathcal{P}_{σ} instead of P_{σ} and with $\delta\sigma$ normalized to make $N'' = 1$.³ (See also [24] for the loop contribution to the spectrum of the axion isocurvature perturbation, given by an identical formula.)

The subscript on the integral reminds us that P_{σ} is set equal to zero at $k < L^{-1}$, cutting out a sphere around each of the singularities. This is necessary, because with \mathcal{P}_{σ} perfectly flat there is a logarithmic divergence whenever the argument of P_{σ} goes to zero, ie. in the infrared. Allowing for slight scale dependence of \mathcal{P}_{σ} , infrared convergence is slow if it occurs at all. In contrast, there is good convergence in the ultra-violet for any reasonable behaviour of \mathcal{P}_{σ} , and the integral will be insensitive to the actual cutoff k_{max} .

It is convenient to define what one might call a reduced bispectrum f_{NL} and a reduced trispectrum τ_{ζ} by

$$B_{\zeta} = \frac{6}{5} f_{\text{NL}} [P_{\zeta}(k_1) P_{\zeta}(k_2) + \text{cyclic}] \quad (33)$$

$$T_{\zeta} = \frac{1}{2} \tau_{\zeta} P_{\zeta}(k_1) P_{\zeta}(k_2) P_{\zeta}(k_{14}) + 23\text{perms.} \quad (34)$$

³ The δN formula was not invoked there, and indeed is irrelevant in the present context. All we really need is that ζ is some quadratic function of a gaussian quantity $\delta\phi$ with zero mean.

At tree-level the reduced quantities are momentum-independent:

$$\frac{6}{5}f_{\text{NL}}^{\text{tree}} = \frac{N''}{N'^2} \quad (35)$$

$$\frac{1}{2}\tau_{\zeta}^{\text{tree}} = \left(\frac{6}{5}f_{\text{NL}}^{\text{tree}}\right)^2. \quad (36)$$

In first order perturbation theory, this definition of f_{NL} coincides with the original one [22] in first-order cosmological perturbation theory, where f_{NL} was defined with respect to the Bardeen potential which was taken to be $\Phi = \frac{3}{5}\zeta$. In that reference it was supposed to be independent of the momenta. Following [23] we will allow momentum dependence, and make the definition without invoking first-order cosmological perturbation theory.⁴ The quantity here denoted as τ_{ζ} was introduced in [8] and denoted as τ_{NL} . Taking them to be momentum-independent observation gives bounds $|f_{\text{NL}}| \lesssim 100$ and $\tau_{\zeta} \lesssim 10^4$, which on cosmological scales are not expected to alter much if there is momentum dependence.

Now I consider estimates of the loop contributions, taking \mathcal{P}_{σ} to be perfectly flat and considering only the physical regime $k_i \gg L^{-1}$. One can arrive at an estimate of $\mathcal{P}_{\zeta}^{\text{loop}}$ by assuming that the integration is dominated by spheres around each of the two singularities, with radii of order k . This gives [8]

$$\mathcal{P}_{\zeta}^{\text{loop}} \simeq 2N''^2 \mathcal{P}_{\sigma}^2 \ln(kL). \quad (37)$$

We deduce that

$$\frac{\mathcal{P}_{\zeta}^{\text{loop}}}{\mathcal{P}_{\zeta}^{\text{tree}}} \sim \frac{N''^2}{N'^4} \mathcal{P}_{\zeta} \ln(kL) \lesssim 10^{-5} \ln(kL), \quad (38)$$

where the inequality comes from the observed spectrum $\mathcal{P}_{\zeta} = (5 \times 10^{-5})^2$ and the observational bound on f_{NL} or τ_{ζ} (by coincidence those bounds give a similar result).

The integration for $\mathcal{P}_{\zeta}^{\text{loop}}$ can actually be done analytically [24] and it happens to give exactly this result. The loop integrals for the bispectrum and trispectrum cannot be done analytically, but they can be estimated in the same way as for $\mathcal{P}_{\zeta}^{\text{loop}}$. Although a more general estimate could be made, we will take all k_i to be of order a common value k for the bispectrum, and all k_i and k_{ij} to be of order a common value for the trispectrum. Then, estimating the loop integrals from the contributions of the singularities as we did for $\mathcal{P}_{\zeta}^{\text{loop}}$, one finds that $(f_{\text{NL}}^{\text{loop}}/f_{\text{NL}}^{\text{tree}})$ and $(\tau_{\zeta}^{\text{loop}}/\tau_{\zeta}^{\text{tree}})$ are both of the same

order as $(\mathcal{P}_{\zeta}^{\text{loop}}/\mathcal{P}_{\zeta}^{\text{tree}})$. Special configurations of the momenta will give additional factors, but observation can probe only a fairly limited range of momenta. Pending further investigation of that issue, we conclude that the loop contributions are very suppressed for a minimal box, and unlikely to be observable.

Our general finding that the loop contributions are negligible relies on the observational bounds on non-gaussianity, and holds because we assumed that $\delta\sigma$ is the only field perturbation contributing to ζ . As we have seen though, in the particular case that σ is the inflaton in a single-field slow-roll inflation model, the loop contribution is very small by virtue of slow-roll, and in consequence the non-gaussianity is very small. With this in mind, we can suppose [8] that the inflaton field perturbation gives the dominant contribution to ζ and write

$$\zeta = \zeta_{\text{inf}} + N'\delta\sigma(\mathbf{x}) + \frac{1}{2}N''\delta\sigma^2(\mathbf{x}) + \dots, \quad (39)$$

where $\delta\sigma$ is say the curvaton. Then the spectrum of ζ is dominated by that of the first term, but its bispectrum and trispectrum might instead be dominated by the third term which would mean that they were dominated by a loop contribution [8]. In that case ζ_{inf} could account for up to 90% of the total, without violating observational bounds on non-gaussianity.

B. Cubic truncation

Now we include the cubic term:

$$\zeta(\mathbf{x}) = N'\delta\sigma(\mathbf{x}) + \frac{1}{2}N''\delta\sigma^2(\mathbf{x}) + \frac{1}{6}N'''\delta\sigma^3(\mathbf{x}). \quad (40)$$

There are no additional tree-level diagrams for the spectrum and bispectrum, but for the trispectrum [13] the cubic truncation gives an additional term

$$T_{\zeta}^{\text{tree3}} = N'^3 N''' P_{\sigma}(k_2)P_{\sigma}(k_3)P_{\sigma}(k_4) + 3 \text{ perms.} \quad (41)$$

The new term cannot cancel the old one since its dependence on the momenta is quite different. To extract optimal information from the observations one should redefine τ_{ζ} and introduce a new quantity g_{ζ} by writing

$$T_{\zeta} = \frac{1}{2}\tau_{\zeta}P_{\zeta}(k_1)P_{\zeta}(k_2)P_{\zeta}(k_4) + 23 \text{ perms.} \\ + g_{\zeta}[P_{\sigma}(k_2)P_{\sigma}(k_3)P_{\sigma}(k_4) + 3 \text{ perms}]. \quad (42)$$

(This g_{ζ} is the same as the g_{NL} of [13] up to a numerical factor.) As with τ_{ζ} it is helpful to allow g_{ζ} to be momentum-dependent because it can then correspond to a (slightly) momentum-dependent loop contribution.

Pending the appropriate observational analysis, we will take $k_i \sim k_{ij} \sim k$ and keep the original definition of τ_{ζ} . Then

$$\tau_{\zeta}^{\text{tree3}} \sim \frac{N'''}{N'^3} \ln(kL) \lesssim 10^4, \quad (43)$$

⁴ We have no need of perturbation theory before horizon entry but it is needed to evolve perturbations afterward. Second order perturbation theory will be needed if $|f_{\text{NL}}| \lesssim 1$, and at that order Φ and ζ are completely different functions. As a result f_{NL} defined with respect to Φ has nothing to do with the f_{NL} of the present paper. Unfortunately, both definitions are in use.

where the final inequality assumes a minimal box and uses the observational bound on τ_ζ which we take to be the same as if τ_ζ were momentum-independent.

Denoting the old contribution by $\tau_\zeta^{\text{tree}2}$ we have the ratios

$$f_{\text{NL}}^{\text{tree}} : \tau_\zeta^{\text{tree}2} : \tau_\zeta^{\text{tree}3} \sim \frac{N''}{N'^2} : \left(\frac{N''}{N'}\right)^2 : \frac{N'''}{N'^3}. \quad (44)$$

As pointed out in [13], $\tau_\zeta^{\text{tree}3}$ might be the first signal of non-gaussianity. This could happen in the curvaton model if the curvaton field evolves strongly after inflation.

It could also in principle happen if σ is the inflaton in a single-component slow-roll inflation model. Slow-roll requires only that all three slow-roll parameters are $\ll 1$, and one might have over a limited range of scales $|\xi|^2 \gg |\eta|$ and $|\xi|^2 \gg \epsilon$. In such a case though, we have to remember that the small non-gaussianity of $\delta\sigma$ at horizon exit will be comparable with the non-gaussianity that we are considering here (ie. that generated by the non-linearity of the δN formula). The known estimates of the bispectrum [11] and trispectrum [12] of $\delta\sigma$ at horizon exit assume that ξ^2 is negligible, and will require modification if it is not. Note also that in such a case the usual [25] formula $n-1 = 2\eta - 6\epsilon$ the spectral index may fail as well [26]. Then one would have to rethink the implication of the current measurement of $n-1$ and of the current bound on $r = 16\epsilon$, for non-gaussianity in slow-roll inflation. Of course, such a rethink is hardly going to alter the conclusion that non-gaussianity in this model will be very hard, if not impossible [27] to detect.

Now we turn to the loop contributions. In the presence of cubic and higher terms, one finds integrations over a single momentum. In the graphical representation, these correspond to loops which start and finish at the same vertex. It has been shown [20] that, instead of including such loops, one can replace the factors N' , N'' and so on by

$$N' \equiv N'(\bar{\sigma}) \rightarrow \langle N'(\mathbf{x}) \rangle \equiv \tilde{N}', \quad (45)$$

and so on.⁵ When these ‘renormalized vertices’ are used, one need only draw ‘renormalized graphs’, which omit all lines corresponding to an integration over a single momentum. We are working at cubic order, which means that only N' gets renormalized:

$$N'(\mathbf{x}) = N' + N''\delta\sigma(\mathbf{x}) + \frac{1}{2}N''' \delta\sigma^2(\mathbf{x}) \quad (46)$$

$$\tilde{N}' = N' + \frac{1}{2}N''' \langle \delta\sigma^2 \rangle, \quad (47)$$

with

$$\langle \delta\sigma^2 \rangle = \int_{L^{-1}}^{k_{\text{max}}} \mathcal{P}_\sigma(k) \frac{dk}{k} \simeq \mathcal{P}_\sigma \ln(k_{\text{max}} L). \quad (48)$$

Let us verify that the loop contributions to the spectrum are still suppressed at cubic order. The renormalized tree-level contribution is

$$\tilde{P}_\sigma^{\text{tree}} = \tilde{N}'^2 P_\sigma(k). \quad (49)$$

Using Eq. (48) and the bound Eq. (43) we find

$$\frac{\tilde{\mathcal{P}}_\zeta^{\text{tree}} - \mathcal{P}_\zeta^{\text{tree}}}{\mathcal{P}_\zeta^{\text{tree}}} \sim \frac{N'''}{N'^3} \mathcal{P}_\zeta \langle \delta\sigma^2 \rangle \lesssim 10^{-5}. \quad (50)$$

With the cubic truncation we have 1- and 2-loop contributions, which have no renormalization. The 1-loop contribution to \mathcal{P}_ζ given by Eq. (28). The 2-loop contribution is [20]

$$P_\zeta^{2\text{-loop}} = \frac{1}{6} \frac{N'''^2}{(2\pi)^6} \int_{L^{-1}} d^3 q_1 d^3 q_2 P(q_1) P(|\mathbf{q}_2 - \mathbf{q}_1|) P(|\mathbf{k} - \mathbf{q}_2|). \quad (51)$$

Taking the integral to be dominated by the three singularities, we estimate

$$P_\zeta^{2\text{-loop}} \sim \frac{N'''^2}{N'^2} \mathcal{P}_\zeta \int_{L^{-1}} d^3 p P_\sigma(p) P_\sigma(|\mathbf{p} - \mathbf{k}|) \quad (52)$$

$$\sim \frac{N'''^2}{N'^2 N''^2} \mathcal{P}_\zeta P_\zeta^{1\text{-loop}} \quad (53)$$

$$\sim \left(\frac{N'''}{N'^3} \mathcal{P}_\zeta \right)^2 P_\zeta^{\text{tree}} \lesssim 10^{-10} P_\zeta^{\text{tree}}. \quad (54)$$

VI. RUNNING

We have advocated the use of a minimal box, but we did see that it might be useful to consider also a super-large box in order to get a handle on $\bar{\sigma}$ within a minimal box. Suppose that for some reason we decide to perform the whole calculation of the correlators in some super-large box with size L . We may compare the outcome of such a calculation with one done in some smaller box with size $M \ll L$, placed within the super-large box. (I am thinking of the size M as being minimal but that is not essential.) An interesting situation then arises, which was explored for the quadratic case in [5, 8].⁶ I now repeat that analysis in a different way, arriving at a differential equation instead of a finite-difference one, and extend it to the cubic truncation. Then I ask how the calculation may be of practical importance.

A. General situation

I think of the super-large box as having a fixed size L . For a calculation within the a smaller box of given

⁵ The authors of [20] use \tilde{N}' to denote $N'(\mathbf{x})$, so that our \tilde{N}' is equal to their $\langle \tilde{N}' \rangle$.

⁶ In [5] the labels L and M are interchanged so that $L < M$. In [8] the labels are as here, but in Eq. (24) of [8] it should be $\log(kM)$ instead of $\log(kL)$.

size M and a given location, $\bar{\sigma}$ of the previous equations becomes $\bar{\sigma}_M$, and Eqs. (3) and (5) become

$$\zeta(\mathbf{x}) = N'_M \delta\sigma_M(\mathbf{x}) + \frac{1}{2} N''_M \delta\sigma_M^2(\mathbf{x}) + \dots, \quad (55)$$

where

$$\delta\sigma_M(\mathbf{x}) = \sigma(\mathbf{x}) - \bar{\sigma}_M, \quad (56)$$

and

$$N'_M = \left. \frac{dN}{d\sigma} \right|_{\bar{\sigma}_M}, \quad N''_M = \left. \frac{d^2 N}{d\sigma^2} \right|_{\bar{\sigma}_M}, \quad (57)$$

and so on. In terms of the original quantities we have

$$\begin{aligned} \delta\sigma_M &= (\bar{\sigma} - \bar{\sigma}_M) + \delta\sigma(\mathbf{x}) \\ N'_M &= N' + N''(\bar{\sigma}_M - \bar{\sigma}) + \frac{1}{2} N'''(\bar{\sigma}_M - \bar{\sigma})^2 + \dots \end{aligned} \quad (58)$$

For a calculation within the smaller box, correlators are defined (in position space) as averages over the smaller box. The spectrum of $\delta\sigma$ is not affected, because $\delta\sigma_M$ differs from $\delta\sigma$ only by a constant. If $\delta\sigma$ is gaussian, we can therefore forget about the change in box size as far as its stochastic properties are concerned. The same is not true of the correlators of ζ though; they will be different because $\zeta(\mathbf{x})$ is a different function, and because the average is taken over a smaller region. Let us denote the spectrum defined within the smaller box by $P_{M\zeta}$ and similarly for the bispectrum and higher correlators.

If we fixed the size and location of the smaller box within the original box, that would be the end of the story. But if the smaller box surrounds the observable Universe, it may be reasonable to suppose that we occupy a typical position within the original box. In that case, instead of considering $P_{M\zeta}$ and so on, one might consider $\langle P_{M\zeta} \rangle$ and so on, the quantities obtained by averaging over the location of the smaller box while keeping its size M fixed. One might hope that this average will give a reasonable estimate of the correlators, evaluated within a smaller box of size M that is fixed at our unknown location.

Since the correlators calculated within a given box can be defined as spatial averages within that box, $\langle P_{M\zeta} \rangle$ and so on must be equal to the quantities P_ζ and so on, evaluated directly within the super-large box. However, if $P_{M\zeta}$ and so on are calculated from Eq. (55) and the spatial average within the super-large is then taken, the separation into tree-level and loop contributions is different. The loop contributions will increase with M , and the tree-level contributions will fall to compensate. As we noticed earlier, the tree-level contribution will usually dominate if the size M is minimal, but that need not remain the case as M is increased to eventually become equal to the super-large box size L . In the following sections we see how the compensation occurs, first for the quadratic truncation and then for the cubic truncation.

The cosmological situation that we have described is analogous to one that occurs in quantum field theory.

There, one also calculates correlators (usually time-order, corresponding to scattering amplitudes) which are the sum of a tree-level and loop contribution. To do the calculation one has to specify a renormalization scale Q . The correlators are independent of Q but the separation into tree-level and loop contributions is not. By choosing Q to be of the same order as the relevant energy scale (set say by the momenta in a scattering process) the tree-level contribution will normally dominate if it is present, otherwise the one-loop contribution will normally dominate and so on. The cosmological situation that we consider is similar, with Q replaced by M . The maximal value L of the super-large box, determined by the amount of slow-roll inflation long before the observable Universe leaves the horizon, provides the infrared cutoff of the theory. Its field theory analogue is the ultra-violet cutoff of the effective field theory, that dictates the maximum choice of the renormalization scale Q .

B. Quadratic truncation

Because Eq. (55) is only quadratic, N'' is just a number and the expectation value with respect to the super-large box is required only for the tree-level terms. We have

$$\langle P_\zeta^{\text{tree}} \rangle = \langle N_M'^2 \rangle P_\sigma(k) \quad (60)$$

$$P_\zeta^{\text{loop}} = \frac{N''^2}{(2\pi)^3} \int_{M^{-1}} d^3 p P_\sigma(p) P_\sigma(|\mathbf{p} - \mathbf{k}|) \quad (61)$$

$$\langle B_\zeta^{\text{tree}} \rangle = 2 \langle N_M'^2 \rangle N'' P_\sigma(k_1) P_\sigma(k_2) + \text{cyclic} \quad (62)$$

$$B_\zeta^{\text{loop}} = \frac{N''^3}{(2\pi)^3} \int_{M^{-1}} d^3 p P_\sigma(p) P_\sigma(p_1) P_\sigma(p_2) \quad (63)$$

$$\langle T_\zeta^{\text{tree}} \rangle = \langle N_M'^2 \rangle N''^2 \mathcal{P}_\zeta(k_1) \mathcal{P}_\zeta(k_2) \mathcal{P}_\zeta(k_{14}) + 23 \text{ perms} \quad (64)$$

$$\begin{aligned} T_\zeta^{\text{loop}} &= \frac{1}{8} \frac{N''^4}{(2\pi)^3} \int_{M^{-1}} d^3 p P_\sigma(p) P_\sigma(p_1) P_\sigma(p_2) P_\sigma(p_{24}) \\ &\quad + 23 \text{ perms.} \end{aligned} \quad (65)$$

Using Eq. (59) we find

$$\langle N_M'^2 \rangle = N'^2 + N''^2 \langle (\bar{\sigma}_M - \bar{\sigma})^2 \rangle. \quad (66)$$

Considered as a function of the position of the box with size M , $\bar{\sigma}_M - \bar{\sigma}$ is simply $\delta\sigma(\mathbf{x})$ smoothed with a top-hat window function. Its mean-square is therefore

$$\langle (\bar{\sigma}_M - \bar{\sigma})^2 \rangle = \int_{L^{-1}}^{M^{-1}} \mathcal{P}_\sigma(k) \frac{dk}{k}. \quad (67)$$

This expression generates M -dependence

$$\frac{d}{d \ln M} \langle (\bar{\sigma}_M - \bar{\sigma})^2 \rangle = -\mathcal{P}_\sigma(M^{-1}). \quad (68)$$

The M -dependence of the tree-level contribution to the spectrum is therefore

$$\frac{d P_{M\zeta}^{\text{tree}}}{d \ln M} = -N''^2 P_\sigma(M^{-1}) P_\sigma(k), \quad (69)$$

and similar expressions hold for the other tree-level contributions.

Now we come to the loop contributions. Because physical momenta have $k \gg L^{-1}$, we can set $P(p) = P(L^{-1})$ near a singularity of the integral at $p = 0$. Differentiating the integral with respect to L , we see that the required cancellation occurs:

$$\frac{d}{d \ln M} P_{\zeta}^{\text{tree}} = -\frac{d}{d \ln M} P_{\zeta}^{\text{loop}}. \quad (70)$$

Similarly, taking into account all of the singularities, one can see that the same is true for the bispectrum and spectrum.

Returning to the analogy with quantum field theory, Eq. (68) is like the ‘running’ of a coupling constant (or other parameter) with the renormalization scale Q . This running makes the correlators of the field theory independent of Q .

C. Cubic truncation

I will just consider the spectrum. Evaluating it in the smaller box and then taking the expectation value with respect to the original box, Eqs. (49)–(51) become

$$\tilde{P}_{\sigma}^{\text{tree}} = \langle \tilde{N}_M'^2 \rangle P_{\sigma}(k) \quad (71)$$

$$\tilde{P}_{\zeta}^{1\text{-loop}} = \frac{1}{2} \frac{\langle N_M''^2 \rangle}{(2\pi)^3} \int_{M^{-1}} d^3 p P_{\sigma}(p) P_{\sigma}(|\mathbf{p} - \mathbf{k}|) \quad (72)$$

$$P_{\zeta}^{2\text{-loop}} = \frac{1}{6} \frac{N_M''^2}{(2\pi)^6} \int_{M^{-1}} d^3 q_1 d^3 q_2 \times P(q_1) P(|\mathbf{q}_2 - \mathbf{q}_1|) P(|\mathbf{k} - \mathbf{q}_2|). \quad (73)$$

The renormalized vertex in the smaller box is

$$\tilde{N}_M' = N_M' + \frac{1}{2} N_M''' \langle \delta \sigma^2 \rangle_M \quad (74)$$

$$\langle \delta \sigma^2 \rangle_M = \int_{M^{-1}}^{k_{\max}} P_{\sigma}(k) \frac{dk}{k}. \quad (75)$$

Using Eq. (59) this gives

$$\langle \tilde{N}_M'^2 \rangle = N'^2 + (N''^2 + N' N''') \langle (\bar{\sigma}_M - \bar{\sigma})^2 \rangle + \frac{1}{4} N_M''^2 \langle \delta \sigma^2 \rangle^2 \quad (76)$$

Only the middle term is M -dependent, giving the running

$$\frac{d}{d \ln M} \langle \tilde{N}_M'^2 \rangle = N''^2 P_{\sigma}(M^{-1}). \quad (77)$$

The running of the renormalized tree-level contribution $\tilde{P}_{M\zeta}^{\text{tree}}$ is therefore simply

$$\frac{d}{d \ln M} \tilde{P}_{M\zeta}^{\text{tree}} = -N'' P_{\sigma}(M^{-1}) P_{\sigma}(k), \quad (78)$$

the same as in the quadratic case.

The running of the prefactor of the renormalized one-loop contribution $\tilde{P}_{M\zeta}^{1\text{-loop}}$ is given by Eq. (59) as

$$\frac{d}{d \ln M} \langle N_M''^2 \rangle = -N_M''^2 P(M^{-1}). \quad (79)$$

Taking into account the running of the integral calculated in the quadratic case this gives

$$\begin{aligned} \frac{d}{dM} \tilde{P}_{\zeta}^{1\text{-loop}}(k) &= -\frac{1}{2} \frac{1}{(2\pi)^3} N_M''^2 P_{\sigma}(M^{-1}) \tilde{P}_{\sigma}^{1\text{-loop}}(k) \\ &+ N'' P_{\sigma}(M^{-1}) P_{\sigma}(k). \end{aligned} \quad (80)$$

Finally, the running of integral in $P_{\zeta}^{2\text{-loop}}$ gives

$$\frac{d}{dM} P_{\zeta}^{2\text{-loop}}(k) = \frac{3}{6} \frac{1}{(2\pi)^3} N_M''^2 P_{\sigma}(M^{-1}) \tilde{P}_{\sigma}^{1\text{-loop}}(k). \quad (81)$$

We see that the total running of P_{ζ} vanishes as required.

D. Application

Does the running have a useful application? At first sight the answer would seem to be ‘yes’, because by going down to a minimal box the loop contributions become negligible. Unfortunately, the gain is illusory because we are not actually calculating correlators within any particular minimal box. Instead we are calculation the expectation values of the correlators within a minimal box (taken within the super-large box). But these are just the actual correlators calculated within the super-large box. As a result, the calculation has all of the uncertainties, and possibly fatal problems, that come with the use of a super-large box.

The problem is that the correlators calculated within a super-large box may be quite different from the ones observed in our Universe. If we throw down a minimal box within the super-large one, the correlators calculated within the minimal box will depend on its location. There is no reason to think that a particular correlator, calculated with the minimal box at our location, will be very close to the result obtained by averaging the position of the minimal box. There is even less reason to think that such will be the case simultaneously for all correlators.

To quantify this concern one would like an estimate of the likely difference between the averaged correlator and the one observed. Extending the terminology coined for the cmb multipoles, one may call that cosmic variance. It will be defined by the correlators evaluated within the super-large box.

As an example we may consider the simplest curvaton model, where the curvaton has a quadratic potential and \dot{H}/H^2 is negligible (see [5] for this case and further references).⁷ In that case, $\bar{\sigma}$ vanishes in a super-large

⁷ Taken literally this case is not realistic because it gives spec-

box and $\bar{\sigma}_M$ is generated entirely by the perturbation $\delta\sigma_M$, having the gaussian probability distribution given by Eq. (24). According to taste, one may fold in this *a priori* expectation with environmental considerations.

VII. CONCLUSION

I have explained how the use of a minimal box leads to fairly clean predictions. In particular I have verified that it makes some specific loop contributions small, by virtue of observational constraints on non-gaussianity.

I have also pointed to some of the uncertainties and index bigger than 1. Keeping \dot{H}/H^2 negligible, the required value $n = 0.95$ could be generated by giving the quadratic potential a bump in the middle. Alternatively one could invoke sig-

possibly fatal problems, that may come with the use of a super-large box. Some of the issues raised here are quite deep and more work needs to be done. Provisionally though, it would seem that the only use of a super-large box is in its possible provision of a probability distribution, for the average of a curvaton-type field within a minimal box.

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nificant \dot{H}/H^2 though the probability distribution of $\bar{\sigma}_M$ would not then be given by Eq. (24).

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